KSU CET UNIT

FIRST YEAR NOTES

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17/8/2019

LINEAR ALGEBRA

MODULE-1

Systems of Linear Equations

A linear system of 'm' equations in 'n' unknowns $x_1, \dots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots &\\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \longrightarrow (1)$$

If all the b_j are zero, then (1) is called a homogeneous system. If at least one b_j is not zero, then (1) is called a non-homogeneous system.

Matrix form of the linear system (1) is

$$Ax = b \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Augmented Matrix ($\tilde{A}$):

The matrix $\tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$

is called the augmented matrix of the system (1).

Gauss' Elimination and Back Substitution

- Convert augmented matrix to a triangular matrix using elementary row transformation.

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19/8/2019

Q: Solve the system of equations by Gauss elimination.

$$2x+3y+2z=5$$
$$3x-5y+5z=2$$
$$3x+9y-z=4$$
$$x+2y+z=3$$

$$\begin{bmatrix} 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

$$Ax = b$$

$$\tilde{A} = \begin{bmatrix} 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \\ 2 & 3 & 2 & 5 \end{bmatrix} \quad R_1 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \\ 0 & -1 & 0 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 3R_1 \\ R_4 \rightarrow R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad R_4 \rightarrow -R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & -4 & -5 \\ 0 & -11 & 2 & -7 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + 11R_2 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_4 \to 2R_4 + R_3$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -8 \\ 0 \end{bmatrix}$$

$$x + 2y + z = 3$$

$$y = 1$$

$$-4z = -8$$

$$z = 2$$

$$\therefore \; x = -1$$

$\therefore$ $x=-1$ $y=1$ $z=2$, is the solution of given system.

Rank of a Matrix

Rank of a matrix is the number of linearly independent rows in a matrix.

(Technique to determine rank: no. of non-zero rows in Echelon form is the rank of that matrix)

David Stel

Linear Algebra

ECHELON FORM:

'Echelon' word meaning is 'step like'.

Conditions of Echelon Form

1. All non-zero rows are lying above all zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All the entries below the leading entry are zeroes

Leading Entry:

First non-zero number in a row is called leading entry.

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Q: By reducing into row echelon form find the rank of the matrix

$$\begin{pmatrix} 3 & 1 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{pmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{bmatrix}_{4\times 5}$$

$$\approx \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{pmatrix} \quad R_1 \longleftrightarrow R_2$$

$$\approx \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & -2 \end{pmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 9R_1 \end{array}$$

$$\approx \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2. \end{array}$$

$$\approx \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_4 \to R_4 - R_3$$

$\therefore$ Rank of the matrix $= 3$

(This is in echelon form. No. of non-zero rows are three. Therefore, rank of the matrix is 3)

[$\rho(A) = 3$
$r(A) = 3$]

CONSISTENCY

Non-Homogeneous System

- If $\rho(\tilde{A}) = \rho(A)$, then the system is consistent.

  * If $\rho(\tilde{A}) = \rho(A) = n$, the system has unique solution (n - no. of unknowns).

  * If $\rho(\tilde{A}) = \rho(A) < n$, the system has infinitely many solutions.

Homogeneous System

- If $\rho(A) = n$, the system has only trivial solutions.
- If $\rho(A) < n$, the system has infinitely many non-trivial solutions.

Q: Test the consistency and hence solve the system of equations:

$$\begin{aligned}
5x+6y+3z &= 2 \\
3x+2y+z &= 2 \\
x+2y+z &= 0 \\
2x-y+2z &= 5 \\
x+3y-z &= -3
\end{aligned}$$

The given system of equation is written in the form $AX = B$.

i.e., $\begin{bmatrix} 5 & 6 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 5 \\ -3 \end{bmatrix}$

We know that the system $AX = B$ is consistent if $\rho(\tilde{A}) = \rho(A)$.

Consider $\tilde{A} = \begin{bmatrix} 5 & 6 & 3 & 2 \\ 3 & 2 & 1 & 2 \\ 1 & 2 & 1 & 0 \\ 2 & -1 & 2 & 5 \\ 1 & 3 & -1 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 5 & 6 & 3 & 2 \\ 2 & -1 & 2 & 5 \\ 1 & 3 & -1 & -3 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & -2 & 2 \\ 0 & -4 & -2 & 2 \\ 0 & -5 & 0 & 5 \\ 0 & 1 & -2 & -3 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 5R_1$, $R_4 \to R_4 - 2R_1$, $R_5 \to R_5 - R_1$

$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & -4 & -2 & 2 \\ 0 & -5 & 0 & 5 \\ 0 & -4 & -2 & 2 \end{bmatrix}$ $R_2 \leftrightarrow R_5$

$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & -10 & -10 \end{bmatrix}$ $R_3 \to R_3 + 4R_2$, $R_4 \to R_4 + 5R_2$, $R_5 \to R_5 + 4R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_4 \to R_4 - R_3 \\ R_5 \to R_5 - R_3 \end{array}$$

$\rho(\tilde{A}) =$ No. of non-zero rows in $\tilde{A} = 3$

$\rho(A) =$ No. of non-zero rows in $A = 3$

$\rho(\tilde{A}) = \rho(A)$

$\therefore$ The given system of solution is consistent.

$n =$ no. of unknown $= 3$

$\rho(\tilde{A}) = \rho(A) = n$

Hence, the system has unique solution.

$$x + 2y + z = 0 \Rightarrow x = 1$$
$$y - 2z = -3 \Rightarrow y = -1$$
$$-10z = -10 \Rightarrow z = 1$$

$\therefore y = -3 + 2 = -1$

$\therefore$ Solution is $x = 1, y = -1, z = 1$

Q: Solve the system of equations:

$$y - 3z = -1$$
$$x + z = 1$$
$$3x + y = 2$$
$$x + y - 2z = 0$$

The given system of equation is written in the form $AX = B$

$$\begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

We know that the system $AX = B$ is consistent if $\rho(\tilde{A}) = \rho(A)$

Consider $\tilde{A} = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2.$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$\rho(\tilde{A}) =$ no. of non-zero rows in $\tilde{A} = 2$

$\rho(A) =$ no. of non-zero rows in $A = 2$.

$\rho(\tilde{A}) = \rho(A)$

$\therefore$ The given system of solution is consistent.

$n =$ no. of unknowns $= 3$

$\rho(\tilde{A}) = \rho(A) < n$

$\therefore$ The system has infinite no. of solution.

$$x + z = 1$$
$$y - 3z = -1$$

Let $z = k$ (k is an arbitrary value)

$$y = -1 + 3k$$
$$x = 1 - k$$

$\therefore$ $x = 1-k$, $y = -1+3k$, $z = k$ is the solution of the given system. $(k \in \mathbb{R})$

Q: Find the values of λ and μ for which the system of equations
$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$
has (i) no solution (ii) unique solution (iii) infinitely many solutions. Find the solutions when they exist.

The given system of equations is written in the form of $AX = B$.

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

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$$\tilde{A} = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}.$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15/2 & -39/2 & -47/2 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - \frac{7}{2}R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right] \quad R_2 \to -2R_2$$

(i) $\lambda = 5;\ \mu \neq 9$

(ii) $\lambda \neq 5$, μ can be any value.

(iii) $\lambda = 5,\ \mu = 9$

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~~Fundamental Theorem~~

Q: Do the equations $x - 3y - 8z = 0$, $3x + y = 0$, $2x + 5y + 6z = 0$ have a non-trivial solution?

The given system is a homogeneous system and it can be written as $AX = 0$

$$\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

We know that, the homogeneous system $AX=0$ has a non-trivial solution, if $\rho(A) < n$.

Here, $\rho(A) < 3$

$$A = \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 11 & 22 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 0 & -22/5 \end{bmatrix} \quad R_3 \to R_3 - \frac{11}{10} R_2$$

But $\rho(A) = 3$

$\rho(A) = n = 3$

$\therefore$ the equ system has only trivial solution.

$$\therefore x = 0,\ y = 0,\ z = 0$$

Hence, the system has no non-trivial solution.

Q: Solve the system of equations :

$$x + 2y - z = 0$$
$$3x + y - z = 0$$
$$2x - y = 0$$

Given system is homogenous and can be written in the form

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & -5 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\rho(A) = 2 < 3$$

Hence, the system has infinitely many non-trivial solution.

$$x + 2y - z = 0 \longrightarrow ①$$
$$-5y + 2z = 0 \longrightarrow ②$$

Let $z = t$

$$② \Rightarrow \quad -5y = -2z$$
$$y = \frac{2}{5} z$$

$$y = \frac{2}{5}t$$

$\therefore$ ① $\Rightarrow$ $x = -2y + z$

$$= -2 \times \frac{2}{5}t + t$$

$$x = -\frac{4}{5}t + t = \frac{t}{5}$$

$$x = \frac{t}{5}$$

$\therefore$ Solution is $x = \frac{t}{5}$, $y = \frac{2}{5}t$, $z = t$,

where, t is any real number.

Matrix Eigen Value Problems

Let $A = [a_{jk}]$ be a given nonzero square matrix of dimension $n \times n$. Consider the following vector equation:

$$Ax = \lambda x \longrightarrow ①$$

The problem of finding nonzero x's and λ's that satisfy the equation is called an eigenvalue problem.

The λ's that satisfy ① are called eigen values of A and the corresponding nonzero x's that also satisfy ① are called eigen vectors of A.

$\boxed{\text{Value of } \lambda \text{ for which (i) has a solution } x \neq 0 \text{ is}}$ called an eigenvalue or characteristic value of the matrix A. The corresponding solutions $x \neq 0$ of (i) are called the eigen vectors or characteristic vectors of A corresponding to that eigenvalue λ.

The set of all the eigenvalues of A is called the spectrum of A. The largest of the absolute values of the eigenvalues of A is called the spectral radius of A.

29/8/2019 Let A be an $n \times n$ square matrix, then, the problem of determining λ and x in the system ~~problem~~ $Ax = \lambda x$ is generally known as Eigen value Problem.

$x = 0$ is always a solution. But in the Eigen value problem, we are considering only the non-zero x.

Note:

We can solve the Eigen value problem $Ax = \lambda x$ by considering

$$Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0$$

which is a homogenous system of n equations in n unknowns.

we know that $(A-\lambda I)x=0$ has a non-trivial solution only if $|A-\lambda I|=0$

Note:

$|A-\lambda I|=0$ is known as the characteristic equation of A, which is a n^{th} degree eqn. in λ. So, let it has n roots, say, $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Now the roots $\lambda_1, \lambda_2, \ldots \lambda_n$ of the characteristic equation $|A-\lambda I|=0$ are known as the Eigen values of A.

$\therefore$ The non zero solution of the homogenous system $(A-\lambda I)x=0$ corresponding to each of the Eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ are known as the Eigen vectors of A.

Q:1) Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Q:2) Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

2/09/2019.

Answers:

1) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$ can be found out by either

$\lambda^3 - (\text{trace of } A)\lambda^2 + \begin{pmatrix} \text{sum of the} \\ \text{cofactor of} \\ \text{the diagonal} \\ \text{elements of } A \end{pmatrix} \lambda - |A| = 0$

OR

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0.$$

$\therefore \lambda^3 - 6\lambda^2 + (4+5+2)\lambda - 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Put $\lambda = 1 \Rightarrow 1 - 6 + 11 - 6 = 0$

$\Rightarrow \lambda = 1$ is a root of $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$.

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

(synthetic division)

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \Rightarrow (\lambda-1)(\lambda-2)(\lambda-3)$

$\therefore$ Roots are $\underline{\underline{1, 2, 3}}$.

$\therefore$ Eigen values are $\underline{\underline{1, 2, 3}}$.

Eigen vector of A corresponding to $\lambda = 1$:

Eigen vector corresponding to λ is

$$(A - \lambda I)X = 0$$

$$(A - I)X = 0$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A - I = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow x_1 + x_2 + x_3 = 0$$

$$-x_3 = 0 \Rightarrow x_3 = 0$$

Put $x_2 = t$

$$x_1 = -t$$

$\therefore x_1 = -t, \; x_2 = t, \; x_3 = 0 \; , \; t \in R$

Put $t = 1$

$\Rightarrow x = -1, \; x_2 = 1, \; x_3 = 0$

$(-1, 1, 0)$

Eigen vector corresponding to $\lambda = 2$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 2 & 2 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$

$$2x_2 - x_3 = 0$$

Put $x_3 = t$

$$x_1 = -x_3 = -t$$

$$x_2 = \frac{x_3}{2} = \frac{t}{2}$$

$\therefore x_1 = -t, \; x_2 = \frac{t}{2}, \; x_3 = t$

Put $t = 2$, (t can be any value)

$x_1 = -2, \; x_2 = 1, \; x_3 = 2$

$(-2, 1, 2)$

Eigen value corresponding to $\lambda = 3$

$$(A - 3I)X = 0$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \end{bmatrix} \quad \begin{aligned} R_2 &\to R_2 + 2R_1 \\ R_3 &\to R_3 - 2R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2.$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$-2x_2 + x_3 = 0$$

Put $x_3 = t$

$$x_2 = \frac{-x_3}{-2} = \frac{t}{2}$$

$$x_1 = x_2 - x_3$$
$$= \frac{t}{2} - t$$
$$= \frac{-t}{2}$$

$\therefore\ x_1 = \frac{-t}{2},\ x_2 = \frac{t}{2},\ x_3 = t$

Put $t=2$,

$x_1 = -1,\ x_2 = 1,\ x_3 = 2$

$(-1, 1, 2)$

2) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$$\lambda^3 - (\text{trace of } A)\lambda^2 + \left(\begin{array}{l}\text{sum of the} \\ \text{cofactor of} \\ \text{diagonal elements} \\ \text{of } A\end{array}\right)\lambda - |A| = 0$$

$$\lambda^3 + \lambda^2 + (-12 + -3 + -6)\lambda - 45 = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

Put $\lambda = 5$

$$125 + 25 - 105 - 45 = 0$$

$\Rightarrow \lambda = 5$ is a root of $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\begin{array}{c|cccc} 5 & 1 & 1 & -21 & -45 \\ & 0 & 5 & 30 & 45 \\ \hline & 1 & 6 & 9 & \boxed{0} \end{array}$$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda+3)^2 = 0$$

$$\lambda = -3, -3$$

$$\therefore \lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow (\lambda-5)(\lambda+3)(\lambda+3)$$

~~Eio~~ ∴ Roots are $5, -3, -3$.

∴ Eigen values are $-3, -3, 5$.

Eigen vector corresponding to λ is

$$(A - \lambda I)x = 0$$

Eigen vector corresponding to $\lambda = -3$.

$$(A + 3I)x = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A + 3I = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 0$$

Let $x_2 = t$

$x_3 = s$

$x_1 = 3x_3 - 2x_2$

$= 3s - 2t$

$\therefore x_1 = 3s - 2t, \; x_2 = t, \; x_3 = s$

$$X = \begin{bmatrix} -2t+3s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 3s \\ 0 \\ s \end{bmatrix}$$

$$= t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 5$.

$(A - 5I)x = 0$

$$A - 5I = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \qquad R_1 \longrightarrow (-1)R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \qquad \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + 7R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} . \qquad R_3' \rightarrow R_3 + 2R_2 .$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 5x_3 = 0$$

$$-8x_2 - 16x_3 = 0$$

Let $x_3 = t$

$$-8x_2 = 16x_3$$

$$x_2 = -2t$$

$$x_1 = -2x_2 - 5x_3$$

$$= 4t - 5t$$

$$= \underline{\underline{-t}}$$

$\therefore x_1 = t, \quad x_2 = -2t \quad x_3 = t$

Put $t = 1$

$x_1 = 1 \quad x_2 = -2 \quad x_3 = 1$

$$X = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

5/9/2019

Q: Find the Eigen values and vectors of

$$A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}.$$

Eigen values are given by

$$|A - \lambda I| = 0.$$

$$\lambda^3 - 3\lambda^2 + [2 + (-1) + 2]\lambda - 1 = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$(\lambda - 1)^3 = 0.$$

$\therefore$ Roots are 1, 1, 1.

Eigen values are 1, 1, 1.

Eigen vector corresponding to $\lambda = 1$

$$(A - \lambda I)X = 0$$

$$(A - I)X = 0$$

$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A-I = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -4 & -7 & -5 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 4R_1 \end{matrix}$$

$-7+8$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2 \quad -5+4$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 0$$

$$-x_2 + x_3 = 0.$$

$$x_3 = x_2$$

Let $x_3 = t$

$\therefore x_2 = t$

$$x_1 = -2x_2 - x_3$$
$$= -2t - t$$
$$= -3t$$

$\therefore x_1 = -3t, \quad x_2 = t, \quad x_3 = t$

Put $t = 1$

$$X = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

DIAGONALISATION:

Let ~~If~~ A be a square matrix, ~~then,~~ we say that A is diagonalisable, if there exist an invertible matrix, P (modal matrix) such that $P^{-1}AP = D$, where, D is a diagonal matrix whose diagonal elements are eigen values of A.

Q: Determine whether the following matrices are diagonalisable or not.

(i) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

(i) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, 2, 3$$

Eigen vector of A corresponding to $\lambda = 1$

$$(A - I)X = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector of A corresponding to $\lambda = 2$.

$$(A - 2I)X = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -t \\ t/2 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Eigen vector of A corresponding to $\lambda=3$

$$(A-3I)X=0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} -t/2 \\ t/2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{\operatorname{adj} P}{|P|}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 0 & 1 & -1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1/2 \\ -2 & -2 & 0 \\ 3 & 3 & 3/2 \end{bmatrix},$$

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & -1/2 \\ -2 & -2 & 0 \\ 3 & 3 & 3/2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D.$$

$P^{-1}AP = D$

$\therefore$ A is <u>diagonalizable</u>.

(ii) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\lambda = -3, -3, 5$$

$$P = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{\operatorname{adj} P}{|P|}$$

$$= \frac{1}{-8}\begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$P^{-1}A = \frac{1}{-8}\begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \frac{1}{-8}\begin{bmatrix} 5 & 10 & -15 \\ -6 & 12 & 18 \\ 3 & 6 & 15 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{-8}\begin{bmatrix} 5 & 10 & -15 \\ -6 & 12 & 18 \\ 3 & 6 & 15 \end{bmatrix}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{-8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$P^{-1}AP = D$$

$\therefore$ A is diagonalizable.

(iii) $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

$$|A-\lambda I| = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$(\lambda-1)^3 = 0$$

$\therefore$ Eigen values are 1, 1, 1.

$$\lambda = 1, 1, 1$$

Eigen vector corresponding to $\lambda = 1$

$$(A-I)X = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -3t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

Since, there is only one independent vector, the given matrix is not diagonalizable.